Eco 5316 Time Series Econometrics

Lecture 5 Autoregressive Moving Average (ARMA) processes

ARMA(p, q) model

- AR or MA models may require a high-order model and thus many parameters to adequately describe the dynamic structure of the data
- Autoregressive Moving-Average (ARMA) models allow to overcome this and allow parsimonious model specification with a small number of parameters

ARMA(p, q) model

- suppose that $\{\varepsilon_t\}$ is a white noise, time series process $\{y_t\}$ follows an ARMA(1,1) if
$$y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$
or equivalently, using the lag operator if $(1-\phi_1 L)y_t = \phi_0 + (1+\theta_1 L)\varepsilon_t$
- more generally, time series process $\{y_t\}$ follows an ARMA(p, q) if
$$y_t = \phi_0 + \sum_{i=1}^{p} \phi_i y_{t-i} + \varepsilon_t + \sum_{i=i}^{q} \theta_q \varepsilon_{t-q}$$
or, using the lag operator
$(1-\phi_1 L - \ldots - \phi_p L^p)y_t = \phi_0 + (1+\theta_1 L + \ldots + \theta_q L^q)\varepsilon_t$

Autocorrelation function for ARMA(p, q) model

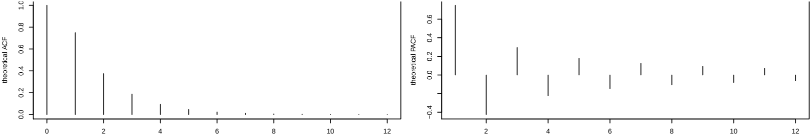
- recall:
  - for AR(p): ACF dies out slowly, PACF drops to zero suddenly after lag p
  - for MA(q): ACF drops to zero immediately after lag q, PACF dies out slowly
- if neither ACF nor PACF drop to zero abruptly we are dealing with and ARMA model
- in this case both ACF and PACF die out slowly in exponential, oscilating exponential of dampened sine wave pattern
- an overview of ACF and PACF for simulated AR(p), MA(q) and ARMA(p, q) models can be found here: https://janduras.shinyapps.io/ARMAsim/lec02ARMAsim.Rmd

Autocorrelation function for ARMA(p, q) model

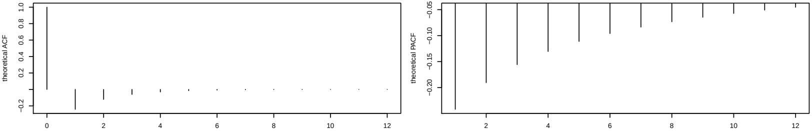
process		ACF	PACF
white noise		$\rho_l = 0$ for all $l > 0$	$\phi_{l,l} = 0$ for all l
AR(1)	$\phi_1 > 0$	exponential decay, $\rho_l = \phi_1^l$	$\phi_{l,l} = \phi_1$, $\phi_{l,l} = 0$ for $l > 1$
	$\phi_1 < 0$	oscillating decay, $\rho_l = \phi_1^l$	$\phi_{l,l} = \phi_1$, $\phi_{l,l} = 0$ for $l > 1$
AR(2)	$\phi_1^2 + 4\phi_2 > 0$	mixture of two exponential decays	$\phi_{1,1} \neq 0$, $\phi_{2,2} \neq 0$, $\phi_{l,l} = 0$ for $l > 2$
	$\phi_1^2 + 4\phi_2 < 0$	dampened sine wave	$\phi_{1,1} \neq 0$, $\phi_{2,2} < 0$, $\phi_{l,l} = 0$ for $l > 2$
AR(p)		decays toward zero in dampened sine wave pattern or oscillating pattern	$\phi_{l,l} = 0$ for $l > p$
MA(1)	$\theta_1 > 0$	$\rho_1 > 0$, $\rho_l = 0$ for all $l > 1$	oscillating decay, $\phi_{1,1} > 0$, $\phi_{2,2} < 0, \ldots$
	$\theta_1 < 0$	$\rho_1 < 0$, $\rho_l = 0$ for all $l > 1$	exponential decay, $\phi_{l,l} < 0$ for all l
MA(2)		$\rho_1 \neq 0$, $\rho_2 \neq 0$, $\rho_l = 0$ for $l > 2$	mixture of two direct or oscillatory exponential decays, or a dampened wave
MA(q)		$\rho_l = 0$ for $l > q$	decays toward zero, may oscillate or have a shape of a dampened sine wave
ARMA(1,1)	$\phi_1 > 0$, $\theta_1 > 0$	exponential decay	oscilating exponential decay
	$\phi_1 > 0$, $\theta_1 < 0$	exponential decay after lag 1	exponential decay
	$\phi_1 < 0$, $\theta_1 > 0$	oscillating exponential decay	oscillating exponential decay
	$\phi_1 < 0$, $\theta_1 < 0$	oscillating exponential decay	exponential decay
ARMA(p, q)		decay (direct or oscillatory) after lag p or dampened sine wave o	decay (direct or oscillatory) after lag q r dampened sine wave

Autocorrelation function for ARMA(p, q) model

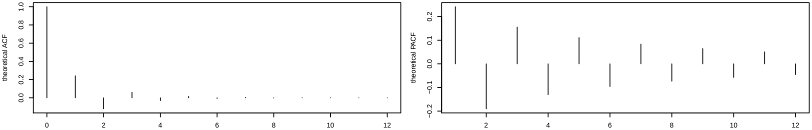
ARMA(1,1) with $\phi_1 = 0.5$, $\theta_1 = 0.9$



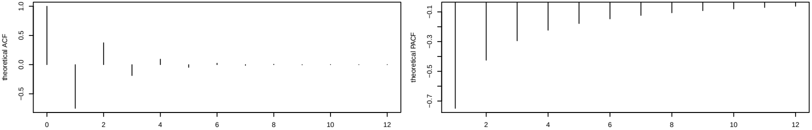
ARMA(1,1) with $\phi_1 = 0.5$, $\theta_1 = -0.9$



ARMA(1,1) with $\phi_1 = -0.5$, $\theta_1 = 0.9$



ARMA(1,1) with $\phi_1 = -0.5$, $\theta_1 = -0.9$



A Couple of Notes

- in practice, we rarely find a data series that precisely conforms to a theoretical ACF or PACF
- consequently, there will be some ambiguities when using the Box-Jenkins methodology
- order (p, q) of an ARMA model may depend on the frequency of the series:
  - daily returns of a market index often show some minor serial correlations
  - monthly returns of the index may not contain any significant serial correlation

Stationarity

- time series $\{y_t\}$ is stationary if it can be represented as a finite order moving average process or a convergent infinite order moving average process
- for an ARMA model to have a convergent MA representation, and thus be stationary, the inverse roots of the polynomial $1 - \phi_1 L - \ldots - \phi_p L^p$ must lie inside the unit circle
- for example, for AR(1) the root of $1 - \phi_1 x = 0$ is $x = \frac{1}{\phi_1}$ its inverse $\omega = \phi_1$ the condition is thus $|\phi_1| < 1$

Invertibility

- time series $\{y_t\}$ is invertible if it can be represented as a finite order autoregressive process or a convergent infinite order autoregressive process
- for an ARMA model to have a convergent AR representation, and thus be invertible, the inverse roots of the polynomial $1+\theta_1 L+\ldots+\theta_q L^q$ must lie inside the unit circle
- for example, for MA(1) the root of $1+\theta_1 x=0$ is $x=-\frac{1}{\theta_1}$ its inverse $\omega=-\theta_1$ the condition is thus $|\theta_1|<1$
- to see why this is necessary note that by repeated substitution

$$\begin{aligned} y_t &= \varepsilon_t+\theta_1\varepsilon_{t-1} && (1)\\ &= \varepsilon_t+\theta_1(y_{t-1}-\theta_1\varepsilon_{t-2}) && (2)\\ &= \varepsilon_t+\theta_1 y_{t-1}-\theta_1^2(y_{t-2}-\theta_1\varepsilon_{t-3}) && (3)\\ &= \varepsilon_t+\theta_1 y_{t-1}-\theta_1^2 y_{t-2}+\theta_1^3(y_{t-3}-\theta_1\varepsilon_{t-4}) && (4)\\ &= \ldots && (5) \end{aligned}$$

we obtain

$$\left(1+\sum_{i=1}^{\infty}(-1)^i\theta_1^i L^i\right)y_t=\varepsilon_t$$

which requires $|\theta_1|<1$

Three Representations for an ARMA Model

1. standard representation as ARMA(p, q)
2. moving average representation of ARMA(p, q)
3. autoregressive representation of ARMA(p, q)

Three Representations for an ARMA Model

1. standard representation as ARMA(p, q)

   compact, useful for estimation, and computing forecasts

   $$\phi(L)y_t = \phi_0 + \theta(L)\varepsilon_t$$

   where $\phi(L) = 1 - \sum_{i=1}^{p} \phi_i L^i$ and $\theta(L) = 1 + \sum_{i=1}^{q} \theta_i L^i$

Three Representations for an ARMA Model

2. moving average representation of ARMA(p, q)

   if all inverse roots of the equation $\phi(L) = 0$ lie inside of the unit circle then $\{y_t\}$ is weakly stationary and can be written as

$$y_t = \frac{\phi_0 + \theta(L)}{\phi(L)} \varepsilon_t \equiv \frac{\phi_0}{\phi(1)} + \psi(L)\varepsilon_t$$

   for AR(1) we for example get

$$y_t = \frac{1}{1 - \phi_1 L}(\phi_0 + \varepsilon_t) = \frac{\phi_0}{1 - \phi_1} + \sum_{l=0}^{\infty} \phi_1^l \varepsilon_{t-l}$$

   coefficients $\{\psi_i\}$ are referred to as the impulse response function of the ARMA model

Three Representations for an ARMA Model

3. autoregressive representation of ARMA(p, q)

   if all roots of the equation $\theta(L) = 0$ lie outside of the unit circle then $\{y_t\}$ is invetible and can be written as

$$\varepsilon_t = \frac{\phi_0 + \phi(L)}{\theta(L)} y_t \equiv \frac{\phi_0}{\theta(1)} + \pi(L) y_t$$

   or equivalently

$$y_t = \frac{\phi_0}{1 + \theta_1 + \ldots + \theta_q} + \sum_{i=l}^{\infty} \pi_l y_{t-l} + \varepsilon_t$$

   coefficients $\{\pi_i\}$ are referred to as π weights of the ARMA model

Example: Total Wages and Salaries in Texas

```r
library(tidyverse)
library(tidyquant)
library(timetk)
library(ggfortify)
library(forecast)

theme_set(theme_bw())
```

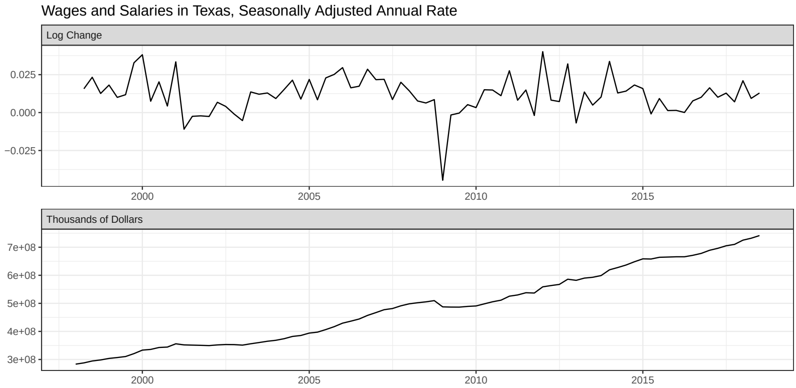
```r
# get quarterly Total Wages and Salaries in Texas, Thousands of Dollars, Seasonally Adjusted Annual Rate
wTX_raw <- tq_get("TXWTOT", get = "economic.data", from = "1980-01-01", to = "2018-12-31")

# note that the sample is quite small
wTX_raw
```

```
## # A tibble: 83 x 2
##    date           price
##    <date>         <int>
##  1 1998-01-01 283350760
##  2 1998-04-01 287832928
##  3 1998-07-01 294612688
##  4 1998-10-01 298346392
##  5 1999-01-01 303809300
##  6 1999-04-01 306873476
##  7 1999-07-01 310509076
##  8 1999-10-01 320868008
##  9 2000-01-01 333337632
## 10 2000-04-01 335838948
## # ... with 73 more rows
```

```r
# log change, a stationary transformation
wTX_tbl <- wTX_raw %>%
    rename(wTX = price) %>%
    mutate(dlwTX = log(wTX) - lag(log(wTX)))
```

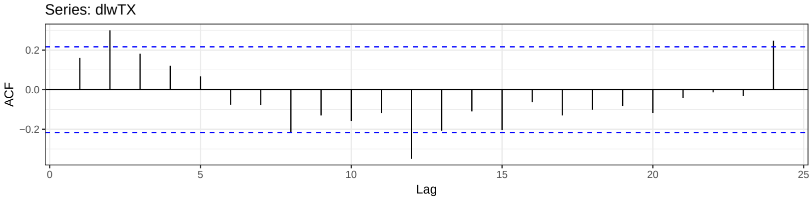
Example: Total Wages and Salaries in Texas

```
wTX_tbl %>%
    tk_xts(date_var = date, select = c("wTX", "dlwTX")) %>%
    autoplot() +
        labs(x = "", y = "",
             title = "Wages and Salaries in Texas, Seasonally Adjusted Annual Rate") +
        facet_wrap(~plot_group, ncol = 1, scales = "free",
                   labeller = labeller(plot_group = c(dlwTX = "Log Change",
                                                      wTX = "Thousands of Dollars"))) +
        theme(strip.text = element_text(hjust = 0))
```

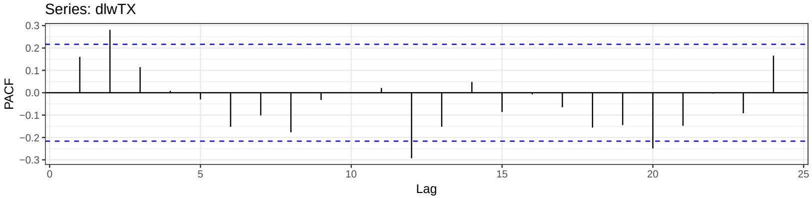


Example: Total Wages and Salaries in Texas

```
dlwTX <- wTX_tbl %>%
    filter(!is.na(dlwTX)) %>%
    tk_xts(date_var = date, select = dlwTX)

nlags <- 24
ggAcf(dlwTX, lag.max = nlags)
```




```
ggPacf(dlwTX, lag.max = nlags)
```



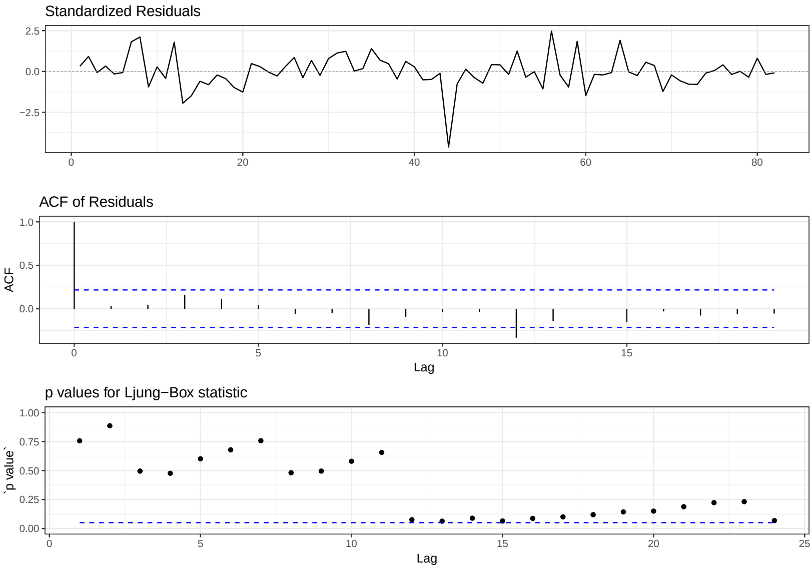
Example: Total Wages and Salaries in Texas

```
m1 <- Arima(dlwTX, order = c(0,0,2))
m1
```

```
## Series: dlwTX
## ARIMA(0,0,2) with non-zero mean
##
## Coefficients:
##          ma1     ma2    mean
##       0.0721  0.2388  0.0118
## s.e.  0.1113  0.0983  0.0017
##
## sigma^2 estimated as 0.0001408:  log likelihood=248.72
## AIC=-489.43   AICc=-488.92   BIC=-479.81
```

Example: Total Wages and Salaries in Texas

```r
ggtsdiag(m1, gof.lag = nlags)
```



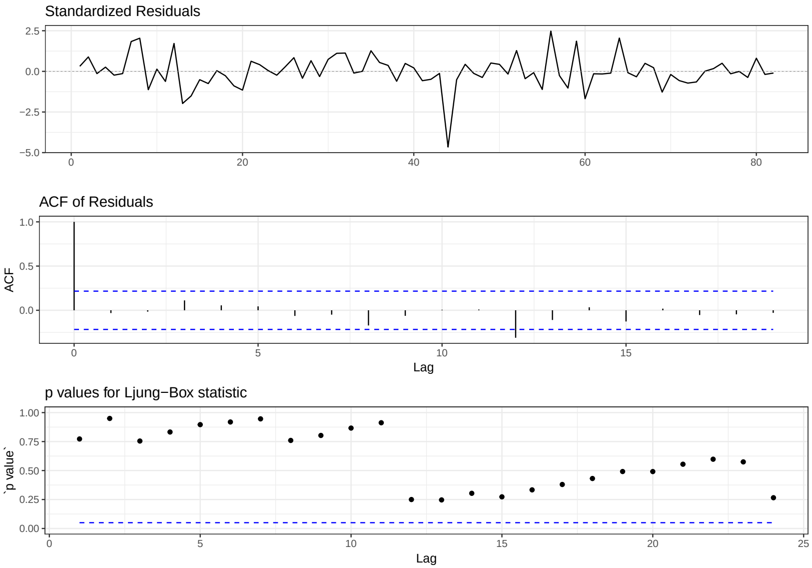
Example: Total Wages and Salaries in Texas

```r
m2 <- Arima(dlwTX, order = c(2,0,0))
m2
```

```
## Series: dlwTX
## ARIMA(2,0,0) with non-zero mean
##
## Coefficients:
##          ar1     ar2    mean
##       0.1143  0.2786  0.0118
## s.e.  0.1048  0.1050  0.0021
##
## sigma^2 estimated as 0.0001373:  log likelihood=249.71
## AIC=-491.42   AICc=-490.9   BIC=-481.8
```

Example: Total Wages and Salaries in Texas

```r
ggtsdiag(m2, gof.lag = nlags)
```



Example: Total Wages and Salaries in Texas

```
# z-statistics for coefficients of AR(2) model - phi1 is not signifficant at any level
m2$coef/sqrt(diag(m2$var.coef))
```

```
##       ar1       ar2 intercept
##  1.090892  2.652768  5.717169
```

```
# p values
(1-pnorm(abs(m2$coef)/sqrt(diag(m2$var.coef))))*2
```

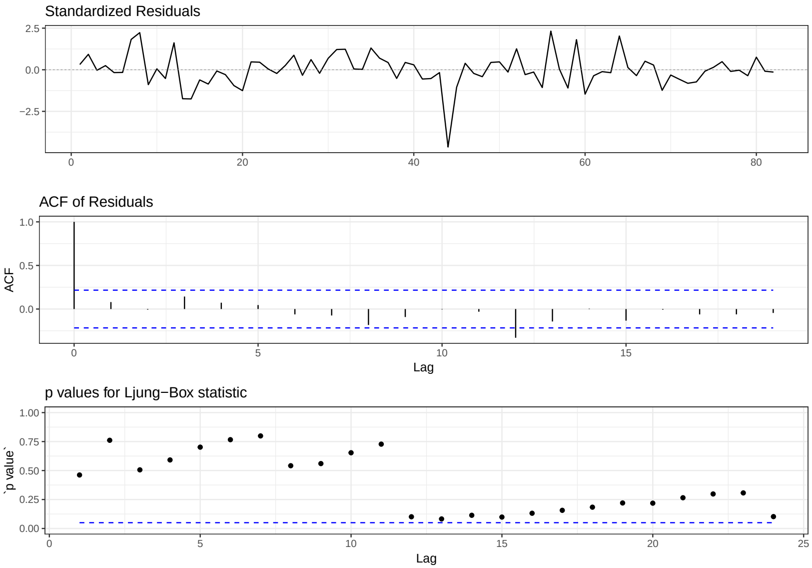
```
##          ar1          ar2    intercept
## 2.753205e-01 7.983480e-03 1.083133e-08
```

```
# estimate ARMA model with a restriction on a parameter
m2.rest <- Arima(dlwTX, order = c(2,0,0), fixed = c(0,NA,NA))
m2.rest
```

```
## Series: dlwTX
## ARIMA(2,0,0) with non-zero mean
##
## Coefficients:
##       ar1     ar2    mean
##         0  0.2965  0.0118
## s.e.    0  0.1045  0.0018
##
## sigma^2 estimated as 0.0001393:  log likelihood=249.12
## AIC=-492.24   AICc=-491.93   BIC=-485.02
```

Example: Total Wages and Salaries in Texas

```
ggtsdiag(m2.rest, gof.lag = nlags)
```



Example: Total Wages and Salaries in Texas

```r
# find the best ARIMA model based on either AIC, AICc or BIC
m3 <- auto.arima(dlwTX, ic="aicc", seasonal=FALSE, stationary=TRUE)
m3
```

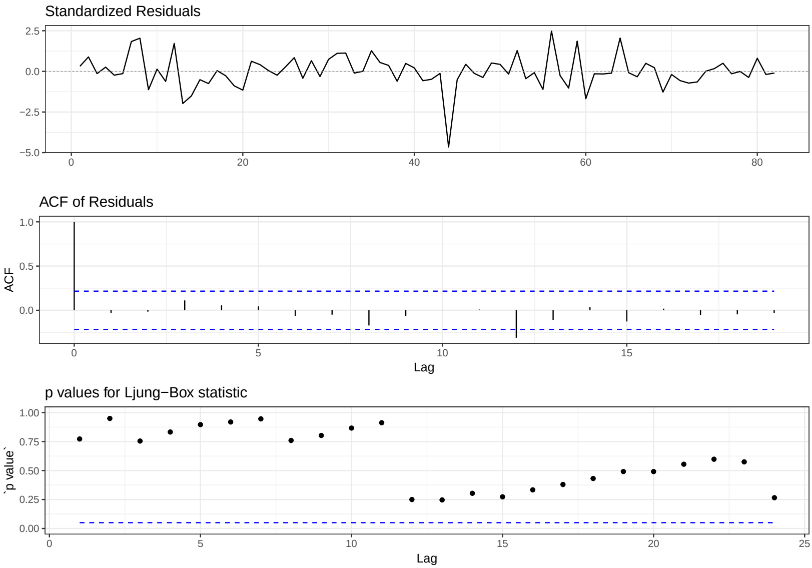
```
## Series: dlwTX
## ARIMA(2,0,0) with non-zero mean
##
## Coefficients:
##          ar1     ar2    mean
##       0.1143  0.2786  0.0118
## s.e.  0.1048  0.1050  0.0021
##
## sigma^2 estimated as 0.0001373:  log likelihood=249.71
## AIC=-491.42   AICc=-490.9   BIC=-481.8
```

```r
# by default auto.arima uses stepwise approach and might end up in a "local minimum" like m3 above
m4 <- auto.arima(dlwTX, ic="aicc", seasonal=FALSE, stationary=TRUE, stepwise=FALSE, approximation=FALSE)
m4
```

```
## Series: dlwTX
## ARIMA(2,0,0) with non-zero mean
##
## Coefficients:
##          ar1     ar2    mean
##       0.1143  0.2786  0.0118
## s.e.  0.1048  0.1050  0.0021
##
## sigma^2 estimated as 0.0001373:  log likelihood=249.71
## AIC=-491.42   AICc=-490.9   BIC=-481.8
```

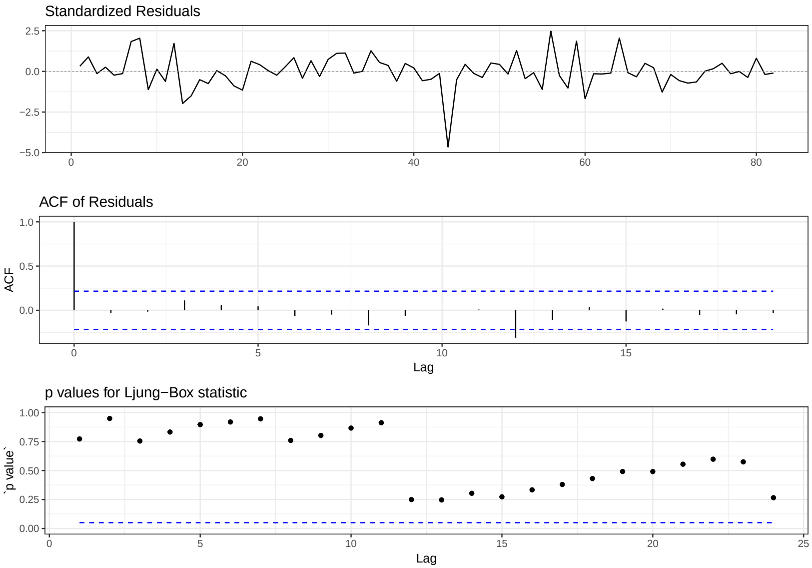
Example: Total Wages and Salaries in Texas

```
ggtsdiag(m3, gof.lag = nlags)
```



Example: Total Wages and Salaries in Texas

```
ggtsdiag(m4, gof.lag = nlags)
```



Example: Total Wages and Salaries in Texas

```
# check staionarity and invertibility of the estimated model - plot inverse AR and MA roots
autoplot(m4)
```